3.4 Verify Properties of Quadrilaterals

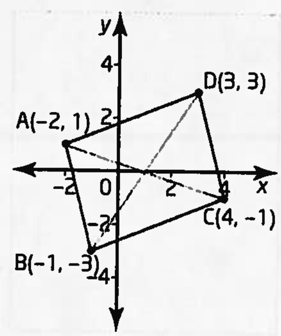
The new addition to the Royal Ontario Museum features a number of quadrilateral panels. This controversial design mimics the shape of mineral crystals.

Investigate

Tools
- grid paper

How can you verify properties of a parallelogram?

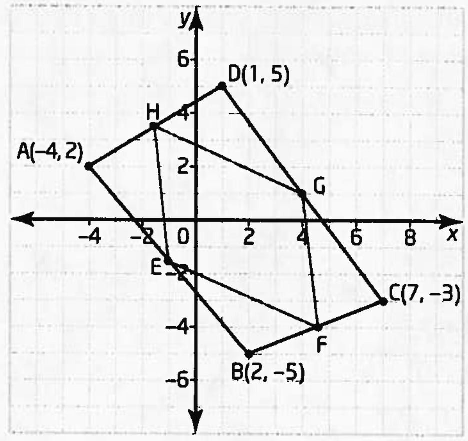
Method 1: Use Pencil and Paper



1. Calculate the slopes of the sides of quadrilateral ABCD. Explain how you can use these slopes to verify that ABCD is a parallelogram.
2. Calculate the length of each side of ABCD. Explain how you can use these lengths to show that ABCD is a parallelogram.
3. Explain how you can use angle measurements to show that ABCD is a parallelogram.
4. Reflect List the properties that you can use to determine whether a given quadrilateral is a parallelogram.
5. Verify that the diagonals, AC and BD, bisect each other.
6. Explain how you can use congruent triangles to verify that the diagonals of ABCD bisect each other.
7. Reflect Are all quadrilaterals with diagonals that bisect each other parallelograms? Can you use this property to determine whether a given quadrilateral is a parallelogram? Justify your answer.

Example 1 Midpoints of a Parallelogram

Verify that quadrilateral EFGH, formed by joining the midpoints of adjacent sides of quadrilateral ABCD, is a parallelogram.



Solution

The simplest way to verify that EFGH is a parallelogram is to show that the slopes of the opposite sides are equal. First, find the coordinates of the midpoint of each side of ABCD.

$$E(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-4 + 2}{2}, \frac{2 + (-5)}{2}\right) = (-1, -1.5)$$

$$F(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{2 + 7}{2}, \frac{-5 + (-3)}{2}\right) = (4.5, -4)$$

$$G(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{7 + 1}{2}, \frac{-3 + 5}{2}\right) = (4, 1)$$

$$H(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{1 + (-4)}{2}, \frac{5 + 2}{2}\right) = (-1.5, 3.5)$$

Use these coordinates to find the slope of each side of EFGH.

$$m_{EF} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - (-1.5)}{4.5 - (-1)} = \frac{-2.5}{5.5} \times \frac{2}{2} = -\frac{5}{11}$$

$$m_{FG} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - (-4)}{4 - 4.5} = \frac{5}{-0.5} = -10$$

$$m_{GH} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3.5 - 1}{-1.5 - 4} = \frac{2.5}{-5.5} \times \frac{2}{2} = -\frac{5}{11}$$

$$m_{HE} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1.5 - 3.5}{-1 - (-1.5)} = \frac{-5}{0.5} = -10$$

Sides EF and GH have the same slope, so they are parallel. Similarly, FG is parallel to EH. Therefore, quadrilateral EFGH is a parallelogram.

Example 2 Properties of a Rhombus

a) Verify that the quadrilateral with vertices P(3, 3), Q(0, 1), R(3, −1), and S(6, 1) is a rhombus.

b) Verify that the diagonals of PQRS bisect each other at right angles.

Solution

a) Find the length of each side of PQRS.

$$\begin{aligned} PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(0 - 3)^2 + (1 - 3)^2} \\ &= \sqrt{(-3)^2 + (-2)^2} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} QR &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - 0)^2 + (-1 - 1)^2} \\ &= \sqrt{3^2 + (-2)^2} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} RS &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(6 - 3)^2 + (1 - (-1))^2} \\ &= \sqrt{3^2 + 2^2} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} PS &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(6 - 3)^2 + (1 - 3)^2} \\ &= \sqrt{3^2 + (-2)^2} \\ &= \sqrt{13} \end{aligned}$$

All four sides are equal in length. Therefore, PQRS is a rhombus.

b) If the diagonals have the same midpoint, they bisect each other.

Find the coordinates of the midpoint of each diagonal.

For PR:

$$\begin{aligned} (x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{3 + 3}{2}, \frac{3 + (-1)}{2}\right) \\ &= (3, 1) \end{aligned}$$

For QS:

$$\begin{aligned} (x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{0 + 6}{2}, \frac{1 + 1}{2}\right) \\ &= (3, 1) \end{aligned}$$

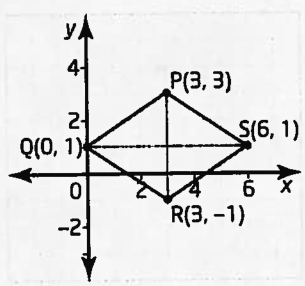
I could show that the diagonals bisect each other by calculating and comparing the lengths PT, QT, RT, and ST.

Since the midpoints of the diagonals have the same coordinates, the diagonals bisect each other.

Now, calculate the slopes of the diagonals.

$$\begin{aligned} m_{PR} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1 - 3}{3 - 3} \\ &= \frac{-4}{0} \end{aligned}$$

$$\begin{aligned} m_{QS} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - 1}{6 - 0} \\ &= 0 \end{aligned}$$



The graph of PQRS confirms that PR is vertical and QS is horizontal.

Since m_{PR} is undefined, PR is a vertical line segment. The zero value for m_{PR} indicates that QS is a horizontal line segment. So, PR and QS are perpendicular.

Therefore, PR and QS bisect each other at right angles.